SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2022 **Trial Examination**

Form VI Mathematics Extension 1

Monday 15th August 2022
12:50pm

General Instructions

- Reading time — 10 minutes
- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 70

Section I (10 marks) Questions 1 – 10
- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (60 marks) Questions 11 – 14
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.
- Place everything inside this question booklet.

Checklist
- Reference sheet
- Multiple-choice answer sheet
- 4 booklets per boy
- Candidature: 131 pupils

Writer: CWG

	Marks
Multiple Choice	
Question 11	
Question 12	
Question 13	
Question 14	
TOTAL	

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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1. Which sum is equal to $\displaystyle\sum_{k=1}^{20}(3k+1)$?

   (A) $1+2+3+\ldots+20$

   (B) $1+4+7+\ldots+61$

   (C) $4+5+6+\ldots+61$

   (D) $4+7+10+\ldots+61$

2. What is the remainder when $P(x) = -6x^3 - 2x^2 + 3x + 10$ is divided by $x+2$?

   (A) -44

   (B) -40

   (C) 40

   (D) 44

3. What is the angle between the vectors $\underset{\sim}{a} = \begin{bmatrix} 14 \\ 5 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$, correct to the nearest degree?

   (A) 23°

   (B) 28°

   (C) 62°

   (D) 67°

4. What is the value of $\displaystyle\lim_{\theta \to 0} \frac{\tan\theta}{2\theta}$?

   (A) 0

   (B) $\dfrac{1}{2}$

   (C) 2

   (D) The limit does not exist.

5. What is the natural domain of the function $f(x) = \sin^{-1}\left(1 - x^2\right)$?

   (A) $-\sqrt{2} \le x \le \sqrt{2}$

   (B) $-1 \le x \le 1$

   (C) $0 \le x \le 1$

   (D) $0 \le x \le \sqrt{2}$

6. What is the exact value of $\displaystyle\int_1^2 \frac{dx}{\sqrt{4 - x^2}}$?

   (A) $\dfrac{\pi}{6}$

   (B) $\dfrac{\pi}{4}$

   (C) $\dfrac{\pi}{3}$

   (D) $\dfrac{\pi}{2}$

7. What is the constant term in the binomial expansion of $\left(4x^2 - \dfrac{1}{x}\right)^9$?

   (A) $-344\,064$

   (B) -5376

   (C) 84

   (D) 5376

8. The displacement, x, of a particle at time $t \ge 0$ is given by

   $x = 7\sin 3t + 24\cos 3t.$

   What is the maximum velocity of the particle?

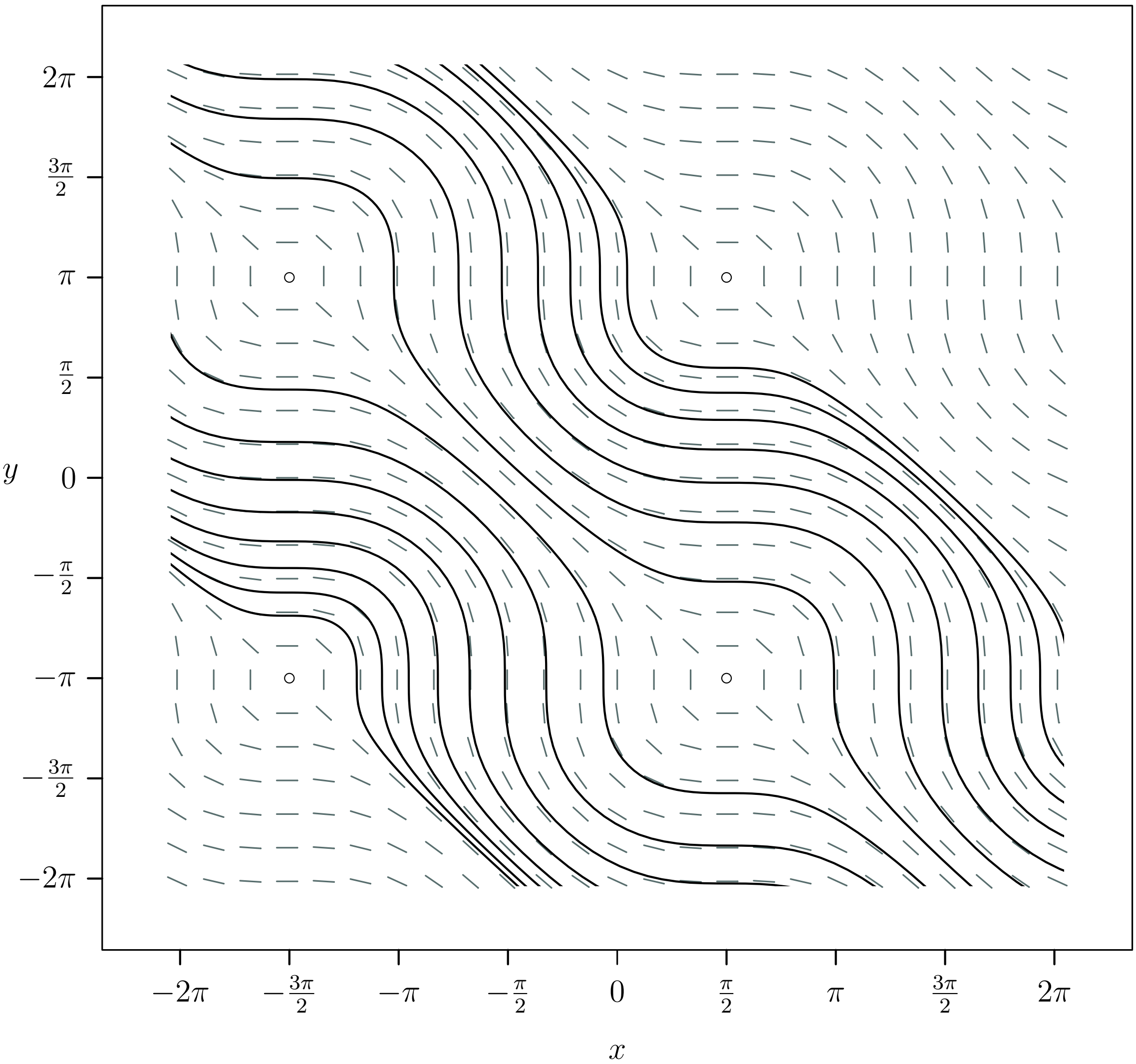
   (A) 25

   (B) 51

   (C) 75

   (D) 225

9. The diagram below shows the direction field (slope field) of a differential equation and some of the solution curves.



Which differential equation best matches this direction field?

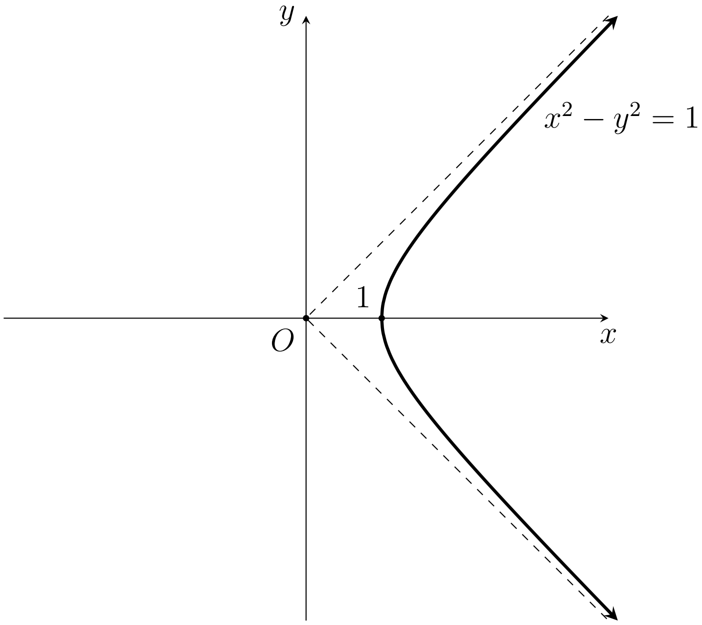
(A) $\dfrac{dy}{dx} = \dfrac{\sin x + 1}{\cos y + 1}$

(B) $\dfrac{dy}{dx} = \dfrac{\sin x + 1}{\cos y - 1}$

(C) $\dfrac{dy}{dx} = \dfrac{\sin x - 1}{\cos y + 1}$

(D) $\dfrac{dy}{dx} = \dfrac{\sin x - 1}{\cos y - 1}$

10.



The diagram above shows the right branch of the hyperbola $x^2 - y^2 = 1$.

Three pairs of parametric equations are listed below:

I $\begin{cases} x = \sec t \\ y = \tan t \end{cases}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

II $\begin{cases} x = \sqrt{1 + t^2} \\ y = t \end{cases}$ for $-\infty < t < \infty$.

III $\begin{cases} x = t \\ y = \sqrt{t^2 - 1} \end{cases}$ for $1 \le t < \infty$.

Which of these pairs of equations give a correct representation of the curve shown in the diagram?

(A) I and II only

(B) I and III only

(C) II and III only

(D) I, II and III

End of Section I

The paper continues in the next section

Section II

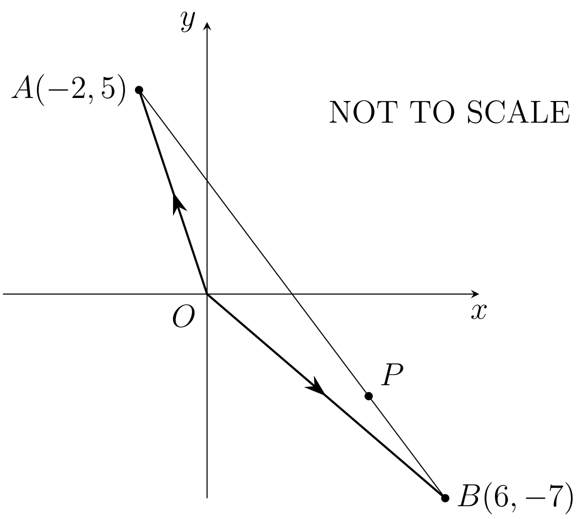
This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet. **Marks**

(a) Solve $\dfrac{x}{x-3} \leq 2$. **3**

(b) Find $\displaystyle\int \sin^2 5x\,dx$. **2**

(c)



The diagram above shows a Cartesian plane where the points A and B have the position vectors $\overrightarrow{OA} = -2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\overrightarrow{OB} = 6\underset{\sim}{i} - 7\underset{\sim}{j}$, respectively, and O is the origin.

(i) Find $\overrightarrow{AB}$ in component form. **1**

(ii) The point P lies on the interval AB with $AP : PB = 3 : 1$. Use vector methods to find the coordinates of P. **2**

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to solve $\sin\theta + 2\cos\theta = 2$ for $0^\circ \leq \theta \leq 360^\circ$. Give your answers correct to the nearest degree. **3**

The question continues on the next page

QUESTION ELEVEN (Continued)

(e) A container of ice cream is stored in a freezer at a temperature of $-18°$C. Before serving, the ice cream is removed from the freezer and left on the kitchen bench at an ambient temperature of $22°$C.

Let T be the temperature of the ice cream in degrees Celsius. Once the ice cream has been removed from the freezer, T increases according to the differential equation

$$\frac{dT}{dt} = k(22 - T),$$

where k is a positive constant and t is the number of minutes after the ice cream was removed from the freezer.

(i) Given that $T = 22 - Ae^{-kt}$ is a solution to the differential equation, determine the value of A. **1**

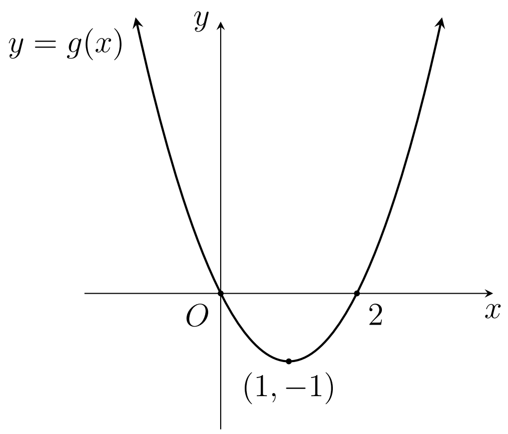
(ii) Twenty minutes after removal from the freezer, the temperature of the ice cream is $-8°$C. Determine the value of k. Give your answer correct to three significant figures. **2**

(iii) It is accepted that ice cream is most palatable when served at $-12°$C. How long should the ice cream have been left on the bench to reach this temperature before serving? Give your answer correct to the nearest minute. **1**

QUESTION TWELVE (15 marks) Start a new answer booklet. **Marks**

(a) How many different ten-letter arrangements can be made using the letters of the word TABLECLOTH? **1**

(b) The diagram below shows the graph of the function $y = g(x)$. This function has zeroes at $x = 0$ and $x = 2$ and a global minimum at $(1, -1)$.

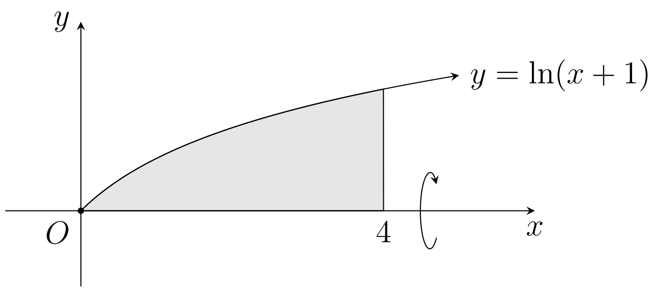


On separate axes, draw neat sketches of the functions listed below. Clearly indicate the location of any asymptotes, any maxima or minima, and any intercepts with the coordinate axes.

(i) $y = \dfrac{1}{g(x)}$ **2**

(ii) $y = g\left(|x|\right)$ **2**

(c) The region under the curve $y = \ln(x + 1)$ for $0 \leq x \leq 4$ is shown in the diagram below. It is rotated about the x-axis to form a solid of revolution.



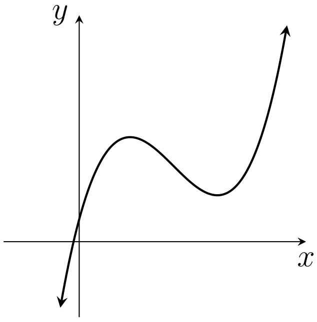
(i) Let V be the volume of the solid of revolution. Write down a definite integral which, if evaluated, would give the exact value of V. **1**

(ii) Use the trapezoidal rule with three function values to estimate V. Give your answer correct to one decimal place. **2**

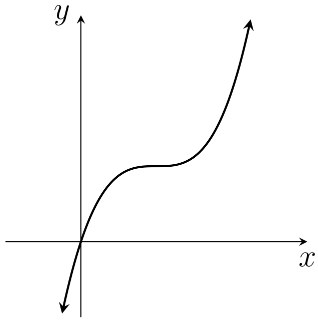
The question continues on the next page

QUESTION TWELVE (Continued)

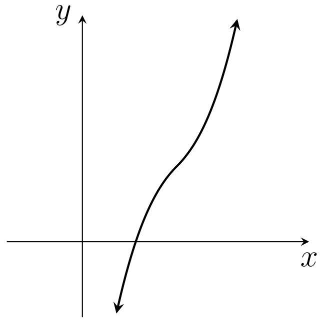
(d) A cubic function can have:



two stationary points;



one stationary point;



or no stationary points.

Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers and $a \neq 0$.

(i) Show that the x-coordinates of the stationary points of $y = f(x)$ satisfy the equation **[2]**

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

(ii) Give an example of a cubic function that has no stationary points. **[1]**

(e) Consider the sequence $T_n = 2n + 2^n$ for integers $n \geq 1$.

(i) Find the first three terms of the sequence. **[1]**

(ii) Use mathematical induction to show that, for $n \geq 1$, **[3]**

$$T_1 + T_2 + \ldots + T_n = n^2 + n - 2 + 2^{n+1}.$$

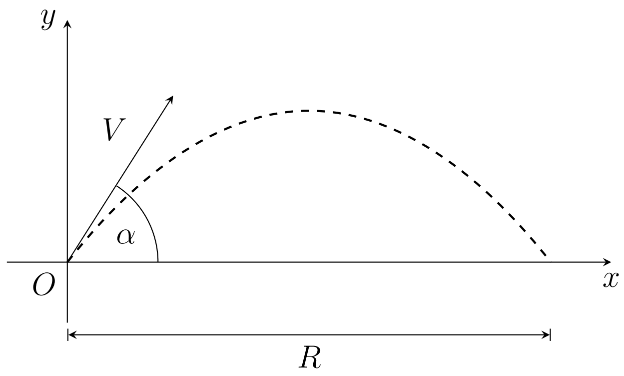
QUESTION THIRTEEN (15 marks) Start a new answer booklet. **Marks**

(a) Find $\dfrac{d}{dx}\left(\cos^{-1}(3x^2)\right)$. 2

(b) Pupils completing Legal Studies for the NSW HSC are awarded an integer mark between zero and 100, inclusive. In the year 2021, a total of 10 935 pupils completed Legal Studies. What is the largest number of pupils that were guaranteed to receive the same mark? 2

(c) The vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular. If $|\underset{\sim}{u}| = 8$ and $|\underset{\sim}{v}| = 3$, evaluate $|\underset{\sim}{u} - 2\underset{\sim}{v}|$. 2

(d) The displacement of a projectile fired from the origin O has a horizontal component x and a vertical component y.



The equations of motion are

$$x = Vt\cos\alpha,$$
$$y = Vt\sin\alpha - 5t^2,$$

where V is the initial speed in metres per second, α is the angle of projection as in the diagram, and t is the time in seconds. The range, R, is the total horizontal distance travelled by the projectile.

(i) Derive an expression for R in terms of V and α. 2

(ii) The maximum range for a given value of V is R_{max}. Show that R_{max} occurs when $\alpha = 45^\circ$ and obtain a fully simplified expression for it. 3

(iii) Show that if the projectile is to hit a target 12 km away, it must be launched at a speed faster than the speed of sound in air, which is 343 m/s. 1

(e) Solve the separable differential equation $\dfrac{dy}{dx} = x(y^2 + 4)$ if the solution passes through the point $(0, 2)$. Express your solution as a function of x. 3

QUESTION FOURTEEN (15 marks) Start a new answer booklet. **Marks**

(a) The mass of a tyrannosaurid dinosaur as a function of its age can be modelled by a logistic equation of the form

$$M(t) = \frac{a}{1 + e^{-b(t-c)}} + k.$$

In this equation:

M is the mass of the dinosaur in kilograms;

t is the age of the dinosaur in years; and

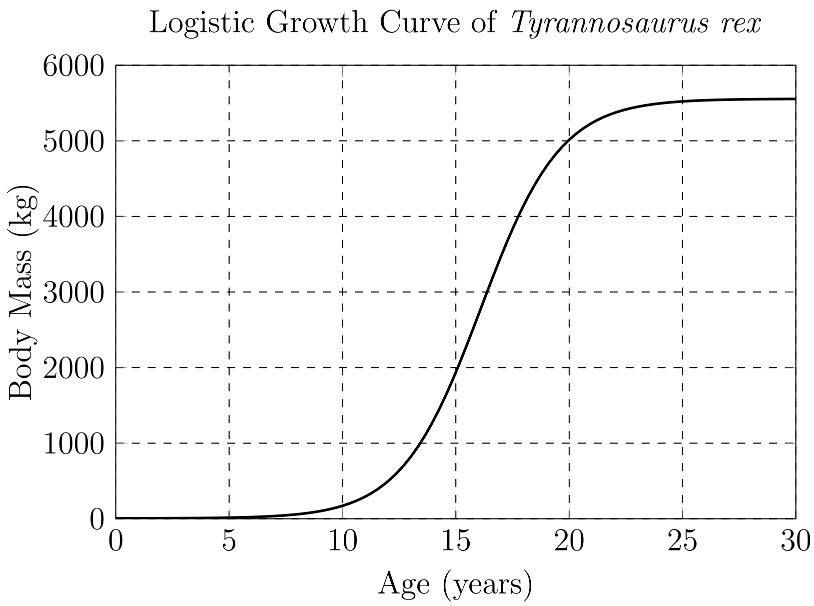
a, b, c and k are positive constants.

For the dinosaur *Tyrannosaurus rex* (*T. rex*), palæontologists have obtained estimates of the constants a, b, c and k, which are shown in the table below.

Constant	Value
a	5 649
b	0·55
c	16·2
k	5

Erickson GM, Makovicky PJ, Currie PJ, Norell MA, Yerby SA, *et al.* (2004) Gigantism and comparative life-history parameters of tyrannosaurid dinosaurs. Nature 430: 772–775.

The diagram below shows the growth curve of *T. rex* for the values given in the table.



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QUESTION FOURTEEN (Continued)

(i) From the diagram of the *T. rex* growth curve, write down an approximate value for the limiting mass. [1]

(ii) Show that the instantaneous growth rate, $G(t) = \dfrac{dM}{dt}$, is given by [1]

$$G(t) = \frac{abe^{-b(t-c)}}{\left[1 + e^{-b(t-c)}\right]^2}.$$

(iii) Show that $G(t)$ has a maximum when $t = c$. [4]

(iv) Hence, or otherwise, determine the maximum growth rate of *T. rex* in kilograms per year predicted by the model. Give your answer correct to the nearest 10 kg/year. [1]

(b) The hyperbolic sine and cosine functions, $\sinh x$ and $\cosh x$, are defined as

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right),$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right).$$

(i) Show that $\dfrac{d}{dx}\sinh x = \cosh x$. [1]

(ii) Hence explain why $f(x) = \sinh x$ has an inverse function defined for all real x. [1]

(iii) Prove the identity $\cosh^2 x - \sinh^2 x = 1$. [1]

(iv) Show that the inverse function of $f(x) = \sinh x$ is given by [3]

$$f^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right).$$

(v) Hence use the substitution $x = \sinh u$ to deduce that [2]

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln\left(x + \sqrt{x^2+1}\right) + C.$$

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Form VI Mathematics Extension 1

Monday 15th August 2022

- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

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Mathematics Extension 1 Trial Solutions

Section I

(1) The first term in the sequence is 4, the last is 61 and the common difference is 3.
Answer: D

(2)

$$\begin{aligned} P(-2) &= -6(-2)^3 - 2(-2)^2 + 3(-2) + 10 \\ &= 48 - 8 - 6 + 10 \\ &= 44 \end{aligned}$$

Answer: D

(3) If θ is the angle between the vectors then

$$\begin{aligned} \cos\theta &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} \\ &= \frac{14 \times 11 + 5 \times 10}{\sqrt{14^2 + 5^2}\sqrt{11^2 + 10^2}} \\ &= \frac{204}{\sqrt{221}\sqrt{221}} \\ &= \frac{204}{221} \\ \therefore \theta &= \cos^{-1}\left(\frac{204}{221}\right) \\ &= 22.61986 \end{aligned}$$

Answer: A

(4)

$$\begin{aligned} \lim_{\theta \to 0} \frac{\tan\theta}{2\theta} &= \frac{1}{2} \lim_{\theta \to 0} \frac{\tan\theta}{\theta} \\ &= \frac{1}{2} \times 1 \end{aligned}$$

Answer: B

(5) The domain is given by

$$\begin{aligned} -1 \leq 1 - x^2 &\leq 1 \\ -2 \leq -x^2 &\leq 0 \\ 0 \leq x^2 &\leq 2 \\ \therefore -\sqrt{2} \leq x &\leq \sqrt{2} \end{aligned}$$

Answer: A

(6)

$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt{4-x^2}} &= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2 \\ &= \sin^{-1} 1 - \sin^{-1}\frac{1}{2} \\ &= \frac{\pi}{2} - \frac{\pi}{6} \\ &= \frac{\pi}{3}\end{aligned}$$

Answer: C

(7) The nth term in the binomial expansion is

$$\begin{aligned}{}^9C_n \left(4x^2\right)^n \left(\frac{-1}{x}\right)^{9-n} &=^9 C_n 4^n \left(-1\right)^{9-n} x^{2n-(9-n)} \\ &=^9 C_n 4^n \left(-1\right)^{9-n} x^{3n-9}\end{aligned}$$

Thus the constant term is when $3n - 9 = 0$. That is, when $n = 3$. In this case, the coefficient is

$$\begin{aligned}{}^9C_3 \times 4^3 \left(-1\right)^6 &= 84 \times 4^3 \\ &= 5376\end{aligned}$$

Answer: D

(8) Using the auxilliary angle method,

$$\begin{aligned}x &= 7\sin 3t + 24\cos 3t \\ &= \sqrt{7^2+24^2}\left(\frac{7}{\sqrt{7^2+24^2}}\sin 3t + \frac{24}{\sqrt{7^2+24^2}}\cos 3t\right) \\ &= 25\left(\frac{7}{25}\sin 3t + \frac{24}{25}\cos 3t\right) \\ &= 25\sin(3t+\theta),\end{aligned}$$

where $\theta = \sin^{-1}\frac{24}{25}$.
Differentiating with respect to time to find the velocity,

$$v = 75\cos(3t+\theta),$$

The maximum velocity is 75.

Answer: C

(9) Note that the tangents are horizontal along the lines $x = \frac{\pi}{2}$ and $x = \frac{-3\pi}{2}$. This is consistent with a numerator of $\sin x - 1$, i.e., (C) or (D). Secondly, the tangents are vertical along the lines $y = \pm\pi$, which is consistent with a denominator of $\cos y + 1$. That is, the answer is (C).

Answer: C

(10) I and II parameterize the curve correctly; III only yields positive y values, and thus only gives the top half of the curve.

Answer: A

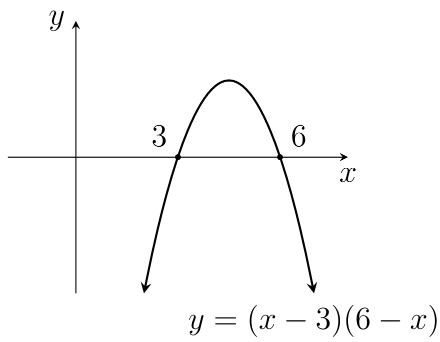
Answer Table	
1	D
2	D
3	A
4	B
5	A
6	C
7	D
8	C
9	C
10	A

Section II

QUESTION ELEVEN (15 Marks)

(a) Firstly note that $x \neq 3$. Multiplying both sides by $(x-3)^2$,

$$\begin{aligned}(x-3)^2 \times \frac{x}{x-3} &\le 2 \times (x-3)^2 \\ x(x-3) &\le 2(x-3)^2 \\ x(x-3) - 2(x-3)^2 &\le 0 \\ (x-3)[x-2(x-3)] &\le 0 \\ (x-3)(6-x) &\le 0\end{aligned}$$



The solution is $x < 3$ or $x \ge 6$.

(b) From the reference sheet,

$$\begin{aligned}\int \sin^2 5x\,dx &= \int \frac{1}{2}(1-\cos 10x)\,dx \\ &= \frac{1}{2}\left(x - \frac{1}{10}\sin 10x\right) + C \\ &= \frac{x}{2} - \frac{\sin 10x}{20} + C\end{aligned}$$

(c) (i)

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (6\underset{\sim}{i} - 7\underset{\sim}{j}) - (-2\underset{\sim}{i} + 5\underset{\sim}{j}) \\ &= 8\underset{\sim}{i} - 12\underset{\sim}{j}\end{aligned}$$

(ii)

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \frac{3}{4}\overrightarrow{AB} \\ &= -2\underset{\sim}{i} + 5\underset{\sim}{j} + \frac{3}{4}(8\underset{\sim}{i} - 12\underset{\sim}{j}) \\ &= 4\underset{\sim}{i} - 4\underset{\sim}{j}\end{aligned}$$

The coordinates of P are $(4, -4)$.

(d) If $t = \tan\frac{\theta}{2}$, then, from the reference sheet, $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$. Thus

$$\begin{aligned}\frac{2t}{1+t^2} + \frac{2(1-t^2)}{1+t^2} &= 2 \\ 2t + 2(1-t^2) &= 2(1+-t^2) \\ 2t + 2 - 2t^2 &= 2 + 2t^2 \\ 4t^2 - 2t &= 0 \\ 2t(2t-1) &= 0\end{aligned}$$

So $t = 0$ or $t = \frac{1}{2}$.

If $t = 0$,

$$\begin{aligned}\frac{\theta}{2} &= 0^\circ, 180^\circ \\ \theta &= 0^\circ, 360^\circ\end{aligned}$$

If $t = \frac{1}{2}$,

$$\begin{aligned}\frac{\theta}{2} &= 26.565\ldots^\circ \\ \theta &= 53^\circ \quad \text{(to nearest degree)}\end{aligned}$$

Thus the solutions are $\theta = 0^\circ, 53^\circ, 360^\circ$.

NOTE: this is the rounding question.

(e) (i) When $t = 0$, $T = -18$, so

$$\begin{aligned}&-18 = 22 - Ae^{0\times k} \\ &\therefore A = 40\end{aligned}$$

(ii) When $t = 20$, $T = -8$, so

$$\begin{aligned}-8 &= 22 - 40e^{-20k} \\ 40e^{-20k} &= 30 \\ e^{-20k} &= 0.75 \\ k &= -\frac{\ln(0.75)}{20} \\ &\approx 0.0144 \text{ (to 3 sig. fig.)}\end{aligned}$$

(iii) If $T = -12$,

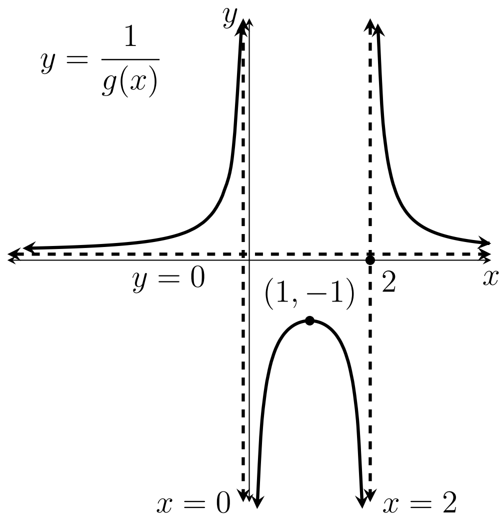
$$\begin{aligned}-12 &= 22 - 40e^{-kt} \\ 40e^{-kt} &= 34 \\ e^{-kt} &= 0.85 \\ t &= -\frac{\ln(0.85)}{k} \\ &\approx 11 \text{ min (to nearest minute.)}\end{aligned}$$

QUESTION TWELVE (15 Marks)

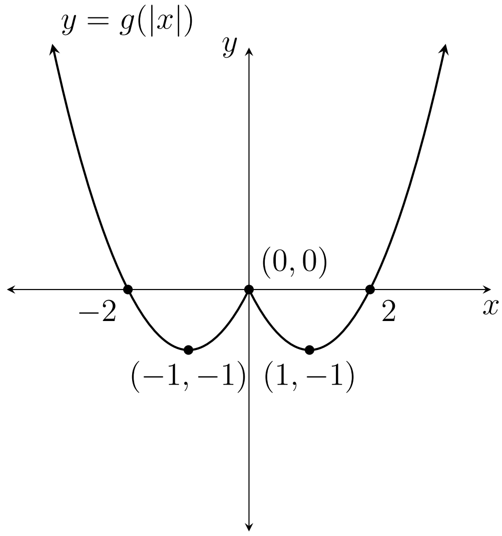
(a) There are two L's and two T's, so the number of arrangements is given by

$$\frac{10!}{2! \times 2!} = 907\,200$$

(b) (i)



(ii)



(c) (i) The integral is

$$V = \int_0^4 \pi \left[\ln(x+1)\right]^2 \, dx$$

(ii) Let $f(x) = \pi\left[\ln(x+1)\right]^2$, and note that $h = 2$.

$$\begin{aligned} V &\approx \frac{h}{2}\left[f(0) + 2f(2) + f(4)\right] \\ &= \frac{2}{2}\left[\pi\left(\ln(1)\right)^2 + 2\pi\left(\ln(3)\right)^2 + \pi\left(\ln(5)\right)^2\right] \\ &= \pi\left[2\left(\ln(3)\right)^2 + \left(\ln(5)\right)^2\right] \\ &= 15.7 \quad \text{(1 d.p.)} \end{aligned}$$

(d) (i)

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ f'(x) &= 3ax^2 + 2bx + c \end{aligned}$$

If $f'(x) = 0$, $3ax^2 + 2bx + c = 0$. Applying the quadratic formula,

$$\begin{aligned} x &= \frac{-2b \pm \sqrt{(2b)^2 - 4(3a)(c)}}{6a} \\ &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\ &= \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a} \\ &= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}, \end{aligned}$$

as required.

(ii) If the cubic has no stationary points then $b^2 - 3ac < 0$. For instance, the values $b = 0$, $a = 1$, $c = 1$ will work and thus the cubic $y = x^3 + x$ has no stationary points.

(e) (i)

$$\begin{aligned} T_1 &= 2(1) + 2^1 = 4 \\ T_2 &= 2(2) + 2^2 = 8 \\ T_3 &= 2(3) + 2^3 = 14 \end{aligned}$$

(ii) RTP: $T_1 + T_2 + T_3 + \cdots + T_n = n^2 + n - 2 + 2^{n+1}$ for $n \geq 1$. $(*)$
For $n = 1$,

$$\begin{aligned} LHS(*) &= T_1 \\ &= 4 \quad \text{from part (i)} \\ RHS(*) &= 1^2 + 1 - 2 + 2^{1+1} \\ &= 4 \end{aligned}$$

Thus $(*)$ is true for $n = 1$.
Suppose that $(*)$ is true for an arbitrary positive integer, $n = k$.

$$T_1 + T_2 + T_3 + \cdots + T_k = k^2 + k - 2 + 2^{k+1}$$

Then, by the inductive hypothesis,

$$T_1 + T_2 + T_3 + \cdots + T_k + T_{k+1} = (k^2 + k - 2 + 2^{k+1}) + T_{k+1}$$

Since $T_{k+1} = 2(k+1) + 2^{k+1}$,

$$\begin{aligned}
T_1 + T_2 + T_3 + \cdots + T_{k+1} &= k^2 + k - 2 + 2^{k+1} + 2(k+1) + 2^{k+1} \\
&= k^2 + k - 2 + 2k + 2 + 2 \times 2^{k+1} \\
&= (k^2 + 2k + 1) + (k+1) - 2 + 2^{(k+1)+1} \\
&= (k+1)^2 + (k+1) - 2 + 2^{(k+1)+1}
\end{aligned}$$

So if $(*)$ is true for k, it is true for $k+1$, as well. Hence the result follows by mathematical induction.

QUESTION THIRTEEN (15 Marks)

(a)

$$\begin{aligned}\frac{d}{dx}\left(\cos^{-1}(3x^2)\right) &= -\frac{1}{\sqrt{1-(3x^2)^2}} \times 6x \\ &= -\frac{6x}{\sqrt{1-9x^4}}\end{aligned}$$

(b) The pupils are the" pigeons" and the possible marks are the "pigeonholes". There are 101 different marks that can be obtained $(0, 1, 2, 3 \ldots, 100)$. Now

$$10\,935 = 108 \times 101 + 27$$

Thus, by the PHP, there must be at least 109 Legal Studies pupils who received the same mark.

(c)

$$\begin{aligned}|\underset{\sim}{u} - 2\underset{\sim}{v}|^2 &= (\underset{\sim}{u} - 2\underset{\sim}{v}) \cdot (\underset{\sim}{u} - 2\underset{\sim}{v}) \\ &= \underset{\sim}{u} \cdot \underset{\sim}{u} - 4\underset{\sim}{u} \cdot \underset{\sim}{v} + 2\underset{\sim}{v} \cdot \underset{\sim}{v}\end{aligned}$$

Since $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular, $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$. Thus

$$\begin{aligned}|\underset{\sim}{u} - 2\underset{\sim}{v}|^2 &= |\underset{\sim}{u}|^2 + 4|\underset{\sim}{v}|^2 \\ &= 64 + 4 \times 9 \\ &= 100 \\ \therefore |\underset{\sim}{u} - 2\underset{\sim}{v}| &= 10\end{aligned}$$

(d) (i) If $y = 0$, then

$$\begin{aligned}Vt\sin\alpha - 5t^2 &= 0 \\ t(V\sin\alpha - 5t) &= 0\end{aligned}$$

So the time of flight is $t = \dfrac{V\sin\alpha}{5}$.

Substiuting $t = \dfrac{V\sin\alpha}{5}$ into x,

$$\begin{aligned}R &= V\cos\alpha \times \frac{V\sin\alpha}{5} \\ &= \frac{V^2\sin\alpha\cos\alpha}{5}\end{aligned}$$

(ii) Using the double angle formula,

$$\begin{aligned}R &= \frac{V^2 \times 2\sin\alpha\cos\alpha}{10} \\ &= \frac{V^2\sin 2\alpha}{10}\end{aligned}$$

The maximum value of R is when $\sin 2\alpha = 1$, which occurs when $2\alpha = 90°$, so $\alpha = 45°$, and $R_{max} = \dfrac{V^2}{10}$.

(iii) If $R_{max} = 12\,000$,

$$\begin{aligned}
12\,000 &= \frac{V^2}{10} \\
V^2 &= 120\,000 \\
\therefore V &= \sqrt{120\,000} \\
&\approx 346.4102 > 343
\end{aligned}$$

(e)

$$\begin{aligned}
\frac{dy}{dx} &= x(y^2+4) \\
\int \frac{1}{y^2+4}\,dy &= \int x\,dx \\
\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) &= \frac{x^2}{2} + C \\
\tan^{-1}\left(\frac{y}{2}\right) &= x^2 + D
\end{aligned}$$

Since $y = 2$ when $x = 0$,

$$\begin{aligned}
\tan^{-1}(1) &= D \\
\therefore D &= \frac{\pi}{4} \\
\tan^{-1}\left(\frac{y}{2}\right) &= x^2 + \frac{\pi}{4} \\
y &= 2\tan\left(x^2 + \frac{\pi}{4}\right)
\end{aligned}$$

QUESTION FOURTEEN (15 Marks)

(a) (i) From the graph, the limiting mass is approximately 5500 kg.

(ii)

$$\begin{aligned} M &= \frac{a}{1+e^{-b(t-c)}} + k \\ &= a\left(1+e^{-b(t-c)}\right)^{-1} + k \\ \therefore G(t) &= \frac{dM}{dt} \\ &= -a(1+e^{-b(t-c)})^{-2} \times -be^{-b(t-c)} \\ &= \frac{abe^{-b(t-c)}}{[1+e^{-b(t-c)}]^2} \end{aligned}$$

(iii)

$$\begin{aligned} G(t) &= \frac{abe^{-b(t-c)}}{[1+e^{-b(t-c)}]^2} \\ \therefore u &= abe^{-b(t-c)}, & u' &= -ab^2e^{-b(t-c)} \\ v &= \left(1+e^{-b(t-c)}\right)^2, & v' &= -2be^{-b(t-c)}\left(1+e^{-b(t-c)}\right) \end{aligned}$$

So

$$\begin{aligned} \frac{dG}{dt} &= \frac{u'v-uv'}{v^2} \\ &= \frac{-ab^2e^{-b(t-c)}(1+e^{-b(t-c)})^2 - abe^{-b(t-c)} \times -2be^{-b(t-c)}\left(1+e^{-b(t-c)}\right)}{[1+e^{-b(t-c)}]^4} \\ &= \frac{\left(1+e^{-b(t-c)}\right)[-ab^2e^{-b(t-c)}(1+e^{-b(t-c)}) + 2ab^2(e^{-b(t-c)})^2]}{[1+e^{-b(t-c)}]^4} \\ &= \frac{ab^2e^{-b(t-c)}[-(1+e^{-b(t-c)}) + 2e^{-b(t-c)}]}{[1+e^{-b(t-c)}]^3} \\ &= \frac{ab^2e^{-b(t-c)}[e^{-b(t-c)}-1]}{[1+e^{-b(t-c)}]^3} \end{aligned}$$

If $\frac{dG}{dt} = 0$, then $e^{-b(t-c)} - 1 = 0$, which implies that $t = c$.
Note that the factors $ab^2e^{-b(t-c)}$ and $[1+e^{-b(t-c)}]^3$ are both positive. Thus the sign of $\frac{dG}{dt}$ changes only if $[e^{-b(t-c)} - 1]$ changes sign.
If $t < c$, then $-b(t-c) > 0$, thus $e^{-b(t-c)} - 1 > 0$ and so $G'(t) > 0$. Similarly, if $t > c$ then $e^{-b(t-c)} - 1 < 0$ and so $G'(t) < 0$.
Hence $G(t)$ has a global maximum when $t = c$.

(iv)

$$\begin{aligned} G(c) &= \frac{abe^{-b(c-c)}}{[1+e^{-b(c-c)}]^2} \\ &= \frac{ab}{4} \\ &= \frac{5649 \times 0.55}{4} \\ &= 776.7375 \\ &\approx 780\,\text{kg/year} \end{aligned}$$

(b) (i)

$$\begin{aligned} \frac{d}{dx}\sinh x &= \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right) \\ &= \frac{1}{2}(e^x - (-1)\times e^{-x}) \\ &= \frac{1}{2}(e^x + e^{-x}) \\ &= \cosh x \end{aligned}$$

(ii) Since $\frac{d}{dx}\sinh x = \cosh x$ the $y = \sinh x$ is increasing for all real x and thus has an inverse function.

(iii)

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\ &= \frac{1}{4}(e^{2x} + 2e^x \times e^{-x} + e^{-2x}) - \frac{1}{4}(e^{2x} - 2e^x \times e^{-x} + e^{-2x}) \\ &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\ &= 1 \end{aligned}$$

(iv) If $x = \sinh y$, then

$$\begin{aligned} x &= \frac{1}{2}(e^y - e^{-y}) \\ 2x &= e^y - \frac{1}{e^y} \end{aligned}$$

Let $e^y = u$, then

$$\begin{aligned} 2x &= u - \frac{1}{u} \\ 2ux &= u^2 - 1 \\ u^2 - 2ux + x^2 &= x^2 + 1 \\ (u-x)^2 &= x^2 + 1 \\ u - x &= \pm\sqrt{x^2+1} \\ u &= x \pm \sqrt{x^2+1} \end{aligned}$$

However, since $u = e^y > 0$, we take the positive square root only, and thus

$$\begin{aligned} e^y &= x + \sqrt{x^2+1} \\ y &= \ln\left(x + \sqrt{x^2+1}\right) \\ \therefore f^{-1}(x) &= \ln\left(x + \sqrt{x^2+1}\right) \end{aligned}$$

NOTE: this can also be proven from the definition of an inverse function, but both parts need to be shown. That is, both $f \circ f^{-1}(x) = x$ and $f^{-1} \circ f(x) = x$ must be clearly demonstrated.

(v) If $x = \sinh u$ then

$$\begin{aligned} \frac{dx}{du} &= \cosh u \\ \therefore dx &= \cosh u \, du \end{aligned}$$

so

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+1}}\, dx &= \int \frac{1}{\sqrt{1+\sinh^2 u}} \times \cosh u \, du \\ &= \int \frac{1}{\sqrt{\cosh^2 u}} \times \cosh u \, du && \text{(from (b)(iii))} \\ &= \int 1 \, du && \text{(Note that } \cosh u > 0 \text{ for all real } u\text{)} \\ &= u + C \end{aligned}$$

If $x = \sinh u$, then $u = \sinh^{-1} x$. From b(iv), $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$. Thus

$$\int \frac{1}{\sqrt{x^2+1}}\, dx = \ln\left(x + \sqrt{x^2+1}\right) + C,$$

as required.